7.3 Loop moving through a nonuniform magnetic field

What if we made a rectangular loop of wire, as shown in Fig. 7.5, and moved it at constant speed through the uniform field **B**? To predict what will happen, we need only ask ourselves – adopting the frame F' – what would happen if we put such a loop into a uniform electric field. Obviously two opposite sides of the rectangle would acquire some charge, but that would be all. Suppose, however, that the field $\mathbf{B}$ in the frame F, though constant in time, is *not uniform* in space. To make this vivid, we show in Fig. 7.6 the field $\mathbf{B}$ with a short solenoid as its source. This solenoid, together with the battery that supplies its constant current, is fixed near the origin in the frame F. (We said earlier there is no electric field in F; if we really use a solenoid of finite resistance to provide the field, there will be an electric field associated with the battery and this circuit. It is irrelevant to our problem and can be ignored. Or we can pack the whole solenoid, with its battery, inside a metal box, making sure the total charge is zero.)

Now, with the loop moving with speed v in the y direction, in the frame F, let its position at some instant t be such that the magnetic field strength is B_1 at the left side of the loop and B_2 along the right side (Fig. 7.6). Let $\mathbf{f}$ denote the force that acts on a charge q that rides along with the loop. This force is a function of position on the loop, at this instant of time. Let's evaluate the line integral of $\mathbf{f}$, taken around the whole loop (counterclockwise as viewed from above). On the two sides of the loop that lie parallel to the direction of motion, $\mathbf{f}$ is perpendicular to the path element $d\mathbf{s}$, so these give nothing. Taking account of the contributions from the other two sides, each of length w, we have

$$\int \mathbf{f} \cdot d\mathbf{s} = qv(B_1 - B_2)w. \tag{7.4}$$

If we imagine a charge q to move all around the loop, in a time short enough so that the position of the loop has not changed appreciably, then Eq. (7.4) gives the work done by the force $\mathbf{f}$. The work done *per unit charge* is $(1/q)\int \mathbf{f} \cdot d\mathbf{s}$. We call this quantity *electromotive force*. We use the symbol $\mathcal{E}$ for it, and often shorten the name to emf. So we have

$$\boxed{\mathcal{E} \equiv \frac{1}{q}\int \mathbf{f} \cdot d\mathbf{s}} \tag{7.5}$$

$\mathcal{E}$ has the same dimensions as electric potential, so the SI unit is the volt, or joule per coulomb. In the Gaussian system, $\mathcal{E}$ is measured in statvolts, or ergs per esu.

We have noted that the force $\mathbf{f}$ does work. However, $\mathbf{f}$ is a magnetic force, and we know that magnetic forces do no work, because the force is always perpendicular to the velocity. So we seem to have an issue here. Is the magnetic force somehow doing work? If not, then what is? This is the subject of Problem 7.2.

The term *electromotive force* was introduced earlier, in Section 4.9. It was defined as the work per unit charge involved in moving a charge around a circuit containing a voltaic cell. We now broaden the definition of emf to include any influence that causes charge to circulate around a closed path. If the path happens to be a physical circuit with resistance R, then the emf $\mathcal{E}$ will cause a current to flow according to Ohm's law: $I = \mathcal{E}/R$. Note that since $\text{curl}\,\mathbf{E} = 0$ for an electrostatic field, such a field cannot cause a charge to circulate around a closed path. By our above definition of electromotive force, an emf must therefore be *nonelectrostatic* in origin. See Varney and Fisher (1980) for a discussion of electromotive force.

In the particular case we are considering, $\mathbf{f}$ is the force that acts on a charge moving in a magnetic field, and $\mathcal{E}$ has the magnitude

$$\mathcal{E} = vw(B_1 - B_2). \tag{7.6}$$

The electromotive force given by Eq. (7.6) is related in a very simple way to the *rate of change of magnetic flux* through the loop. (We will be quantitative about this in Theorem 7.1.) By the magnetic flux through a loop we mean the surface integral of $\mathbf{B}$ over a surface that has the loop for its boundary. The flux Φ through the closed curve or loop C in Fig. 7.7(a) is given by the surface integral of $\mathbf{B}$ over S_1:

$$\Phi_{S_1} = \int_{S_1} \mathbf{B} \cdot d\mathbf{a}_1. \tag{7.7}$$

We could draw infinitely many surfaces bounded by C. Figure 7.7(b) shows another one, S_2. Why don't we have to specify which surface to use in computing the flux? It *doesn't make any difference* because $\int \mathbf{B} \cdot d\mathbf{a}$ will have the same value for all surfaces. Let's take a minute to settle this point once and for all.

The flux through S_2 will be $\int_{S_2} \mathbf{B} \cdot d\mathbf{a}_2$. Note that we let the vector $d\mathbf{a}_2$ stick out from the upper side of S_2, to be consistent with our choice of side of S_1. This will give a positive number if the net flux through C is upward:

$$\Phi_{S_2} = \int_{S_2} \mathbf{B} \cdot d\mathbf{a}_2. \tag{7.8}$$

We learned in Section 6.2 that the magnetic field has zero divergence: $\text{div}\,\mathbf{B} = 0$. It follows then from Gauss's theorem that, if S is any *closed* surface ("balloon") and V is the volume inside it, we have

$$\int_S \mathbf{B} \cdot d\mathbf{a} = \int_V \text{div}\,\mathbf{B}\, dv = 0. \tag{7.9}$$

Apply this to the closed surface, rather like a kettledrum, formed by joining our S_1 to S_2, as in Fig. 7.7(c). On S_2 the outward normal is *opposite* the vector $d\mathbf{a}_2$ we used in calculating the flux through C. Thus

$$0 = \int_S \mathbf{B} \cdot d\mathbf{a} = \int_{S_1} \mathbf{B} \cdot d\mathbf{a}_1 + \int_{S_2} \mathbf{B} \cdot (-d\mathbf{a}_2), \tag{7.10}$$

or

$$\int_{S_1} \mathbf{B} \cdot d\mathbf{a}_1 = \int_{S_2} \mathbf{B} \cdot d\mathbf{a}_2. \tag{7.11}$$

This shows that it doesn't matter which surface we use to compute the flux through C.

This is all pretty obvious if you realize that div $\mathbf{B} = 0$ implies a kind of spatial conservation of flux. As much flux enters any volume as leaves it. (We are considering the situation in the whole space at one instant of time.) It is often helpful to visualize "tubes" of flux. A flux tube (Fig. 7.8) is a surface at every point on which the magnetic field line lies in the plane of the surface. It is a surface through which no flux passes, and we can think of it as containing a certain amount of flux, as a fiber optic cable contains fibers. Through any closed curve drawn tightly around a flux tube, the same flux passes. This could be said about the electric field $\mathbf{E}$ only for regions where there is no electric charge, since div $\mathbf{E} = \rho/\epsilon_0$. The magnetic field always has zero divergence everywhere.

Returning now to the moving rectangular loop, let us find the *rate of change* of flux through the loop. In time dt the loop moves a distance $v\,dt$. This changes in two ways the total flux through the loop, which is $\int \mathbf{B} \cdot d\mathbf{a}$ over a surface spanning the loop. As you can see in Fig. 7.9, flux is gained at the right, in amount $B_2 wv\,dt$, while an amount of flux $B_1 wv\,dt$ is lost at the left. Hence $d\Phi$, the change in flux through the loop in time dt, is

$$d\Phi = -(B_1 - B_2)wv\,dt. \tag{7.12}$$

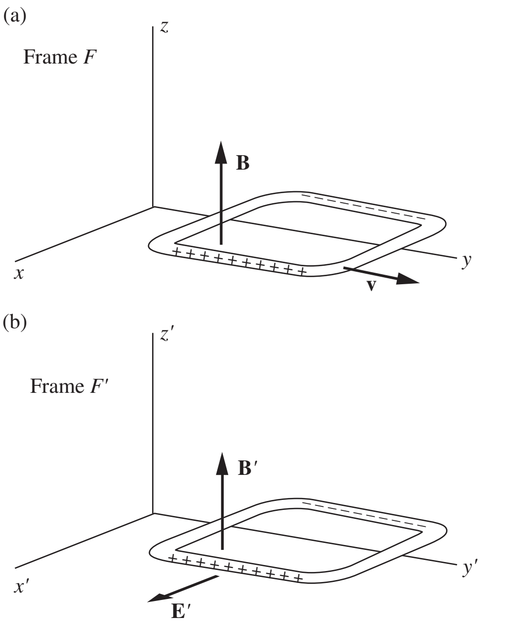


Figure 7.5.
(a) Here the wire loop is moving in a uniform magnetic field $\mathbf{B}$. (b) Observed in the frame F' in which the loop is at rest, the fields are $\mathbf{B}'$ and $\mathbf{E}'$.

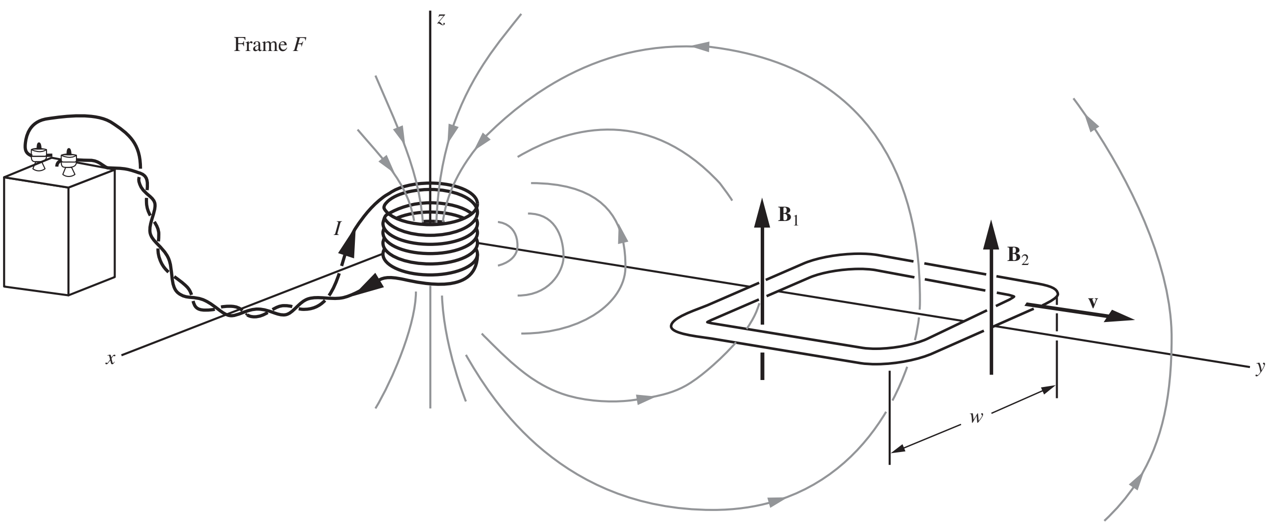


Figure 7.6.
Here the field $\mathbf{B}$, observed in F, is not uniform. It varies in both direction and magnitude from place to place.

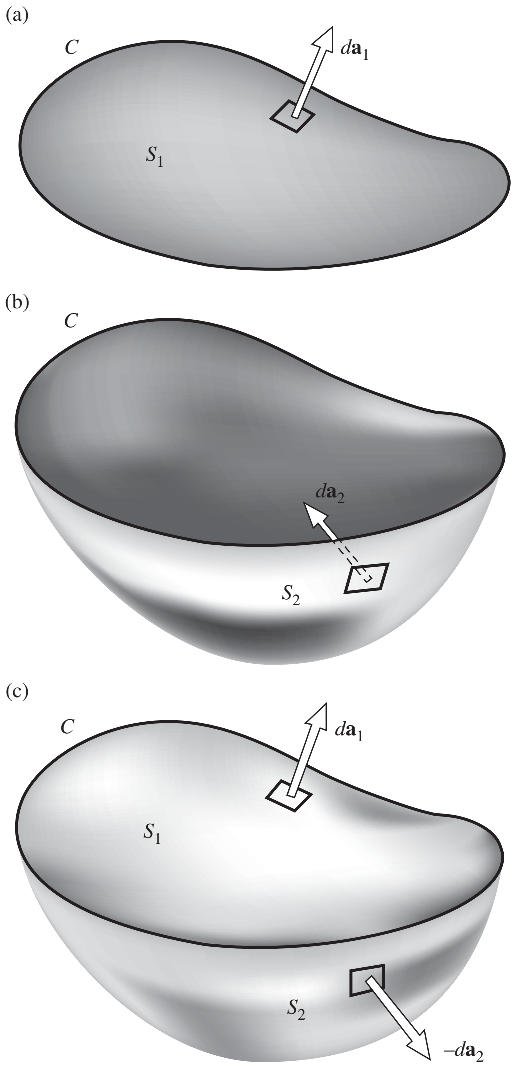


Figure 7.7.
(a) The flux through C is $\Phi = \int_{S_1} \mathbf{B} \cdot d\mathbf{a}_1$. (b) S_2 is another surface that has C as its boundary. This will do just as well for computing Φ.
(c) Combining S_1 and S_2 to make a closed surface, for which $\int \mathbf{B} \cdot d\mathbf{a}$ must vanish, proves that $\int_{S_1} \mathbf{B} \cdot d\mathbf{a}_1 = \int_{S_2} \mathbf{B} \cdot d\mathbf{a}_2$.

Comparing Eq. (7.12) with Eq. (7.6), we see that, in this case at least, the electromotive force can be expressed as $\mathcal{E} = -d\Phi/dt$. It turns out that this is a general result, as the following theorem states.

Theorem 7.1 *If the magnetic field in a given frame is constant in time, then for a loop of any shape moving in any manner, the emf $\mathcal{E}$ around the loop is related to the magnetic flux Φ through the loop by*

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}} \tag{7.13}$$

Proof The loop C in Fig. 7.10 occupies the position C_1 at time t, and it is moving so that it occupies the position C_2 at time $t + dt$. A particular element of the loop ds has been transported with velocity $\mathbf{v}$ to its new position. S indicates a surface that spans the loop at time t. The flux through the loop at this instant of time is

$$\Phi(t) = \int_S \mathbf{B} \cdot d\mathbf{a}. \tag{7.14}$$

The magnetic field $\mathbf{B}$ comes from sources that are stationary in our frame of reference and remains constant in time, at any point fixed in this frame. At time $t + dt$ a surface that spans the loop is the original surface S, left fixed in space, augmented by the "rim" dS. (Remember, we are allowed to use *any* surface spanning the loop to compute the flux through it.) Thus

$$\Phi(t + dt) = \int_{S+dS} \mathbf{B} \cdot d\mathbf{a} = \Phi(t) + \int_{dS} \mathbf{B} \cdot d\mathbf{a}. \tag{7.15}$$

Hence the change in flux, in time dt, is just the flux $\int_{dS} \mathbf{B} \cdot d\mathbf{a}$ through the rim dS. On the rim, an element of surface area $d\mathbf{a}$ can be expressed as $(\mathbf{v}\, dt) \times d\mathbf{s}$, because this cross product has magnitude $|\mathbf{v}\, dt||d\mathbf{s}| \sin\theta$ and points in the direction perpendicular to both $\mathbf{v}\, dt$ and $d\mathbf{s}$; the $\sin\theta$ in the magnitude gives the correct area of the little parallelogram in Fig. 7.10. So the integral over the surface dS can be written as an integral around the path C, in this way:

$$d\Phi = \int_{dS} \mathbf{B} \cdot d\mathbf{a} = \int_C \mathbf{B} \cdot [(\mathbf{v}\, dt) \times d\mathbf{s}]. \tag{7.16}$$

Since dt is a constant for the integration, we can factor it out to obtain

$$\frac{d\Phi}{dt} = \int_C \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{s}). \tag{7.17}$$

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ of any three vectors satisfies the relation $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}$, which you can verify by explicitly writing out each side in Cartesian components. Using this identity to rearrange the integrand in Eq. (7.17), we have

$$\frac{d\Phi}{dt} = -\int_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}. \tag{7.18}$$

Now, the force on a charge q that is carried along by the loop is just $q\mathbf{v} \times \mathbf{B}$, so the electromotive force, which is the line integral around the loop of the force per unit charge, is just

$$\mathcal{E} = \int_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}. \tag{7.19}$$

Comparing Eq. (7.18) with Eq. (7.19), we get the simple relation given in Eq. (7.13), valid for arbitrary shape and motion of the loop. (We did not even have to assume that $\mathbf{v}$ is the same for all parts of the loop!) In summary, the line integral around a moving loop of $\mathbf{f}/q$, the force per unit charge, is just the negative of the rate of change of flux through the loop. □

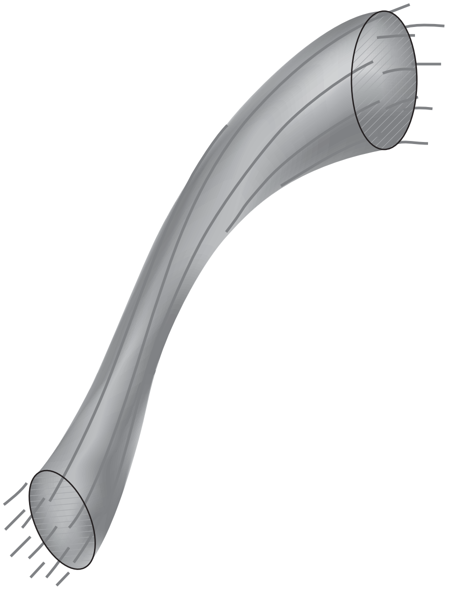


Figure 7.8.
A flux tube. Magnetic field lines lie in the surface of the tube. The tube encloses a certain amount of flux Φ. No matter where you chop it, you will find that $\int \mathbf{B} \cdot d\mathbf{a}$ over the section has the same value Φ. A flux tube doesn't have to be round. You can start somewhere with any cross section, and the course of the field lines will determine how the section changes size and shape as you go along the tube.

The sense of the line integral and the direction in which flux is called positive are to be related by a right-hand-thread rule. For instance, in Fig. 7.6, the flux is *upward* through the loop and is *decreasing*. Taking the minus sign in Eq. (7.13) into account, our rule would predict an electromotive force that would tend to drive a positive charge around the loop in a counterclockwise direction, as seen looking down on the loop (Fig. 7.11).

There is a better way to look at this question of sign and direction. Note that if a current should flow in the direction of the induced electromotive force, in the situation shown in Fig. 7.11, this current itself would create some flux through the loop in a direction to *counteract* the assumed flux change (because the Biot–Savart law, Eq. (6.49), tells us that the contributions from this current to the $\mathbf{B}$ field inside the loop all point upward in Fig. 7.11). That is an essential physical fact, and not the consequence of an arbitrary convention about signs and directions. It is a manifestation of the tendency of systems to resist change. In this context it is traditionally called *Lenz's law.*

Lenz's law *The direction of the induced electromotive force is such that the induced current creates a magnetic field that opposes the change in flux.*

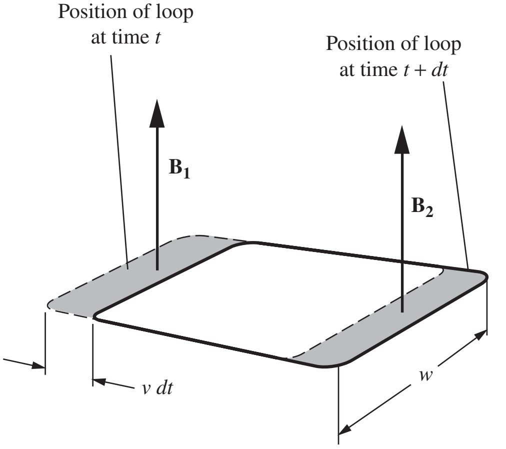


Figure 7.9.
In the interval dt, the loop gains an increment of flux $B_2 wv\,dt$ and loses an increment $B_1 wv\,dt$.

Another example of Lenz's law is illustrated in Fig. 7.12. The conducting ring is falling in the magnetic field of the coil. The flux through the ring is *downward* and is *increasing* in magnitude. To counteract this change, some new flux upward is needed. It would take a current flowing around the ring in the direction of the arrows to produce such flux. Lenz's law assures us that the induced emf will be in the correct direction to cause such a current.

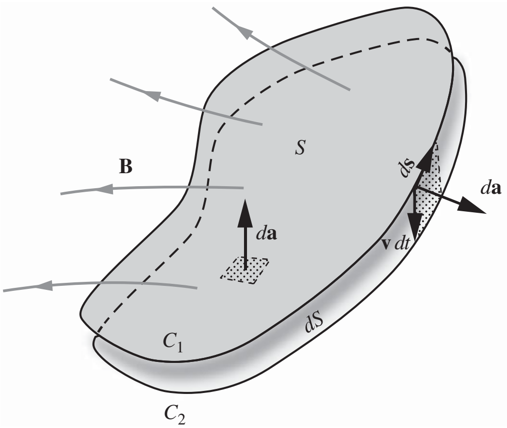


Figure 7.10.
The loop moves from position C_1 to position C_2 in time dt.

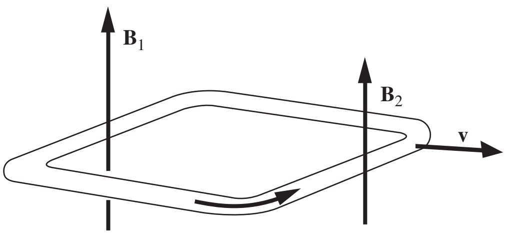


Figure 7.11.
The flux through the loop is upward and is decreasing in magnitude as time goes on. The arrow shows the direction of the electromotive force, that is, the direction in which positive charge tends to be driven.

If the electromotive force causes current to flow in the loop that is shown in Figs. 7.6 and 7.11, as it will if the loop has a finite resistance, some energy will be dissipated in the wire. What supplies this energy? To answer that, consider the force that acts on the current in the loop if it flows in the sense indicated by the arrow in Fig 7.11. The side on the right, in the field B_2, will experience a force toward the right, while the opposite side of the loop, in the field B_1, will be pushed toward the left. But B_1 is greater than B_2, so the net force on the loop is toward the left, *opposing the motion.* To keep the loop moving at constant speed, some external agency has to do work, and the energy thus invested eventually shows up as heat in the wire (see Exercise 7.30). Imagine what would happen if Lenz's law were violated, or if the force on the loop were to act in a direction to assist the motion of the loop!

Example (Sinusoidal $\mathcal{E}$) A very common element in electrical machinery and electrical instruments is a loop or coil that rotates in a magnetic field. Let's apply what we have just learned to the system shown in Fig. 7.13, a single loop rotating at constant speed in a magnetic field that is approximately uniform. The mechanical essentials, shaft, bearings, drive, etc., are not drawn. The field $\mathbf{B}$ is provided by the two fixed coils. Suppose the loop rotates with angular velocity ω, in radians/second. If its position at any instant is specified by the angle θ, then $\theta = \omega t + \alpha$, where the constant α is simply the position of the loop at $t = 0$. The component of $\mathbf{B}$ perpendicular to the plane of the loop is $B \sin\theta$. Therefore the flux through the loop at time t is

$$\Phi(t) = SB \sin(\omega t + \alpha), \tag{7.20}$$

where S is the area of the loop. For the induced electromotive force we then have

$$\mathcal{E} = -\frac{d\Phi}{dt} = -SB\omega \cos(\omega t + \alpha). \tag{7.21}$$

If the loop instead of being closed is connected through slip rings to external wires, as shown in Fig. 7.13, we can detect at these terminals a sinusoidally alternating potential difference.

A numerical example will show how the units work out. Suppose the area of the loop in Fig. 7.13 is 80 cm^2, the field strength B is 50 gauss, and the loop is rotating at 30 revolutions per second. Then $\omega = 2\pi \cdot 30$, or 188 radians/second. The amplitude, that is, the maximum magnitude of the oscillating electromotive force induced in the loop, is

$$\mathcal{E}_0 = SB\omega = \left(0.008\,\text{m}^2\right)\left(0.005\,\text{tesla}\right)\left(188\,\text{s}^{-1}\right) = 7.52 \cdot 10^{-3}\ \text{V}. \tag{7.22}$$

You should verify that 1 $\text{m}^2 \cdot$ tesla/s is indeed equivalent to 1 volt.

7.5 Universal law of induction

Let's carry out three experiments with the apparatus shown in Fig. 7.15. The tables are on wheels so that they can be easily moved. A sensitive galvanometer has been connected to our old rectangular loop, and to increase any induced electromotive force we put several turns of wire in the loop rather than one. Frankly though, our sensitivity might still be marginal, with the feeble source of magnetic field pictured. Perhaps you can devise a more practical version of the experiment.

Experiment I. With constant current in the coil and table 1 stationary, table 2 moves toward the right (away from table 1) with speed v. The *galvanometer deflects.* We are not surprised; we have already analyzed this situation in Section 7.3.

Experiment II. With constant current in the coil and table 2 stationary, table 1 moves to the left (away from table 2) with speed v. The *galvanometer deflects.* This doesn't surprise us either. We have just discussed in Section 7.4 the equivalence of Experiments I and II, an equivalence that is an example of Lorentz invariance or, for the low speeds of our tables, Galilean invariance. We know that in both experiments the deflection of the galvanometer can be related to the rate of change of flux of $\mathbf{B}$ through the loop.

Experiment III. Both tables remain at rest, but we vary the current I in the coil by sliding the contact K along the resistance strip. We do this in such a way that the *rate of decrease* of the field $\mathbf{B}$ at the loop is the same as it was in Experiments I and II. *Does the galvanometer deflect?*

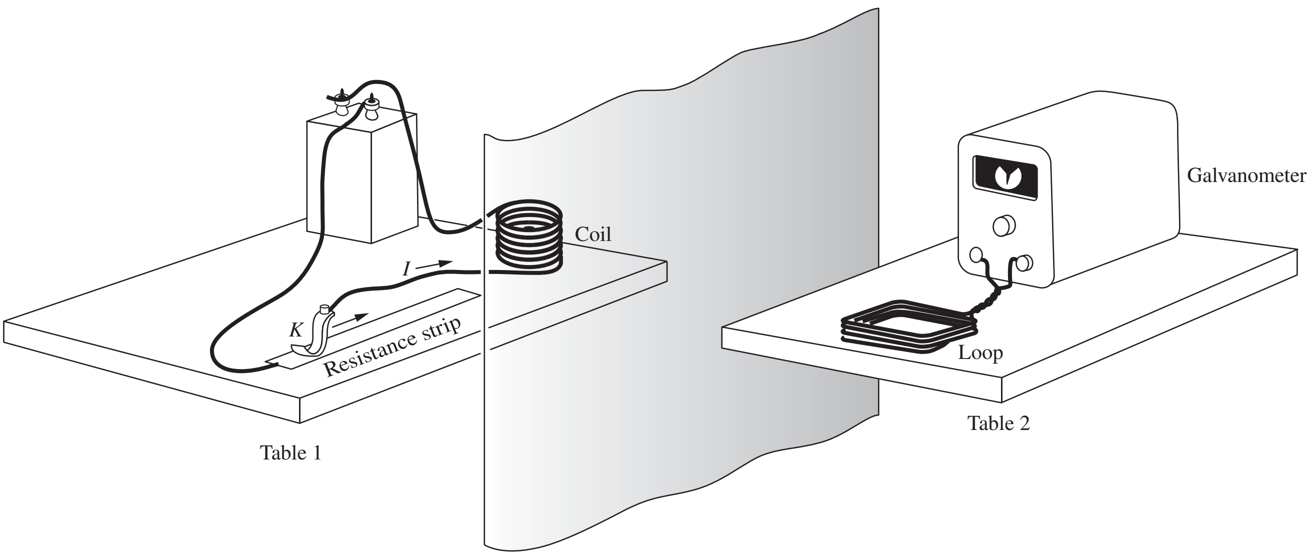


Figure 7.15.
We imagine that either table can move or, with both tables fixed, the current I in the coil can be gradually changed.

For an observer stationed at the loop on table 2 and measuring the magnetic field in that neighborhood as a function of time and position, there is no way to distinguish among Experiments I, II, and III. Imagine a black cloth curtain between the two tables. Although there might be minor differences between the field configurations for II and III, an observer who did not know what was behind the curtain could not decide, on the basis of local **B** measurements alone, which case it was. Therefore if the galvanometer did *not* respond with the same deflection in Experiment III, it would mean that the relation between the magnetic and electric fields in a region depends on the nature of a remote source. Two magnetic fields essentially similar in their local properties would have associated electric fields with different values of $\int \mathbf{E} \cdot d\mathbf{s}$.

We find by experiment that III *is* equivalent to I and II. The galvanometer deflects, by the same amount as before. Faraday's experiments were the first to demonstrate this fundamental fact. The electromotive force we observe depends only on the rate of change of the flux of **B**, and not on anything else. We can state as a universal relation *Faraday's law of induction:*

If C is some closed curve, stationary in coordinates x, y, z; if S is a surface spanning C; and if $\mathbf{B}(x, y, z, t)$ is the magnetic field measured in x, y, z, at any time t, then

$$\mathcal{E} = \int_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a} = -\frac{d\Phi}{dt} \qquad \text{(Faraday's law)} \tag{7.26}$$

Using the vector derivative curl, we can express this law in differential form. If the relation

$$\int_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a} \tag{7.27}$$

is true for *any* curve C and spanning surface S, as our law asserts, it follows that, at any point,

$$\text{curl}\, \mathbf{E} = -\frac{d\mathbf{B}}{dt}. \tag{7.28}$$

To show that Eq. (7.28) follows from Eq. (7.27), we proceed as usual to let C shrink down around a point, which we take to be a nonsingular point for the function **B**. Then in the limit the variation of **B** over the small patch of surface **a** that spans C will be negligible and the surface integral will approach simply $\mathbf{B} \cdot \mathbf{a}$. By definition (see Eq. (2.80)), the limit approached by $\int_C \mathbf{E} \cdot d\mathbf{s}$ as the patch shrinks is $\mathbf{a} \cdot \text{curl}\, \mathbf{E}$. Thus Eq. (7.27) becomes, in the limit,

$$\mathbf{a} \cdot \text{curl}\, \mathbf{E} = -\frac{d}{dt}(\mathbf{B} \cdot \mathbf{a}) = \mathbf{a} \cdot \left(-\frac{d\mathbf{B}}{dt}\right). \tag{7.29}$$

Since this holds for *any* infinitesimal **a**, it must be that[1]

$$\text{curl}\,\mathbf{E} = -\frac{d\mathbf{B}}{dt}. \tag{7.30}$$

Recognizing that **B** may depend on position as well as time, we write $\partial\mathbf{B}/\partial t$ in place of $d\mathbf{B}/dt$. We have then these two entirely equivalent statements of the law of induction:

$$\boxed{\begin{aligned} \int_C \mathbf{E}\cdot d\mathbf{s} &= -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{a} \\ \text{curl}\,\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \end{aligned}} \tag{7.31}$$

With Faraday's law of induction, we are one step closer to the complete set of Maxwell's equations. We will obtain the last piece to the puzzle in Chapter 9.

In Eq. (7.31) the electric field **E** is to be expressed in our SI units of volts/meter, with **B** in teslas, $d\mathbf{s}$ in meters, and $d\mathbf{a}$ in m^2. The electromotive force $\mathcal{E} = \int_C \mathbf{E}\cdot d\mathbf{s}$ will then be given in volts. In Gaussian units the relation expressed by Eq. (7.31) looks like this:

$$\int_C \mathbf{E}\cdot d\mathbf{s} = -\frac{1}{c}\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{a},$$

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}. \tag{7.32}$$

Here **E** is in statvolts/cm, **B** is in gauss, $d\mathbf{s}$ and $d\mathbf{a}$ are in cm and cm^2, respectively, and c is in cm/s. The electromotive force $\mathcal{E} = \int_C \mathbf{E}\cdot d\mathbf{s}$ will be given in statvolts.

The magnetic flux Φ, which is $\int_C \mathbf{B}\cdot d\mathbf{a}$, is expressed in tesla-$\text{m}^2$ in our SI units, and in gauss-cm^2, a unit exactly 10^8 times smaller, in Gaussian units (because $1\ \text{m}^2 = 10^4\ \text{cm}^2$ and $1\ \text{tesla} = 10^4$ gauss, exactly). The SI flux unit is assigned a name of its own, the *weber*.

When in doubt about the units, you may find one of the following equivalent statements helpful:

- Electromotive force in statvolts equals:
  $1/c$ times rate of change of flux in gauss-cm^2/s.
- Electromotive force in volts equals:
  rate of change of flux in tesla-m^2/s.
- Electromotive force in volts equals:
  10^{-8} times rate of change of flux in gauss-cm^2/s.

[1] If that isn't obvious, note that choosing **a** in the x direction will establish that $(\text{curl}\,\mathbf{E})_x = -dB_x/dt$, and so on.

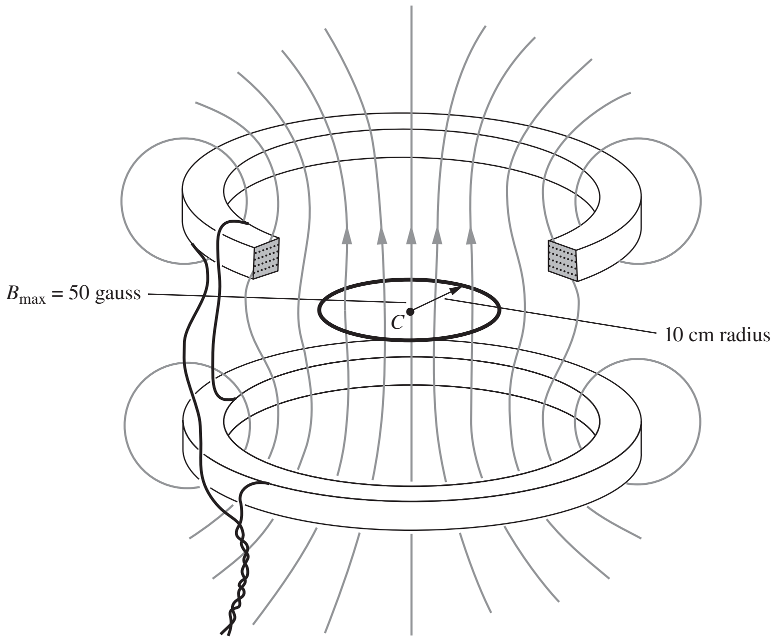


Figure 7.16.
Alternating current in the coils produces a magnetic field which, at the center, oscillates between 50 gauss upward and 50 gauss downward. At any instant the field is approximately uniform within the circle C.

If these seem confusing, don't try to remember them. Just remember that you can look them up on this page.

The differential expression, $\text{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t$, brings out rather plainly the point we tried to make earlier about the local nature of the field relations. The variation in time of $\mathbf{B}$ in a neighborhood completely determines curl $\mathbf{E}$ there – nothing else matters. That does not completely determine $\mathbf{E}$ itself, of course. Without affecting this relation, any electrostatic field with $\text{curl}\,\mathbf{E} = 0$ could be superposed.

Example (Sinusoidal B field) As a concrete example of Faraday's law, suppose coils like those in Fig. 7.13 are supplied with 60 cycles per second alternating current, instead of direct current. The current and the magnetic field vary as $\sin(2\pi \cdot 60\,\text{s}^{-1} \cdot t)$, or $\sin(377\,\text{s}^{-1} \cdot t)$. Suppose the amplitude of the current is such that the magnetic field $\mathbf{B}$ in the central region reaches a maximum value of 50 gauss, or 0.005 tesla. We want to investigate the induced electric field, and the electromotive force, on the circular path 10 cm in radius shown in Fig. 7.16. We may assume that the field B is practically uniform in the interior of this circle, at any instant of time. So we have

$$B = (0.005\ \text{T})\sin(377\,\text{s}^{-1} \cdot t). \tag{7.33}$$

The flux through the loop C is

$$\begin{aligned}\Phi &= \pi r^2 B = \pi \cdot (0.1\ \text{m})^2 \cdot (0.005\ \text{T})\sin(377\,\text{s}^{-1} \cdot t)\\ &= 1.57 \cdot 10^{-4}\sin(377\,\text{s}^{-1} \cdot t)\ \text{T}\,\text{m}^2.\end{aligned} \tag{7.34}$$

Using Eq. (7.26) to calculate the electromotive force, we obtain

$$\mathcal{E} = -\frac{d\Phi}{dt} = -(377\,\text{s}^{-1}) \cdot 1.57 \cdot 10^{-4} \cos(377\,\text{s}^{-1} \cdot t)\ \text{T m}^2$$
$$= -0.059 \cos(377\,\text{s}^{-1} \cdot t)\ \ \text{V}. \tag{7.35}$$

The maximum attained by $\mathcal{E}$ is 59 millivolts. The minus sign will ensure that Lenz's law is respected, if we have defined our directions consistently. The variation of both Φ and $\mathcal{E}$ with time is shown in Fig. 7.17.

What about the electric field itself? Usually we cannot deduce **E** from a knowledge of curl **E** alone. However, our path C is here a circle around the center of a symmetrical system. *If there are no other* electric fields around, we may assume that, on the circle C, **E** lies in that plane and has a constant magnitude. Then it is a trivial matter to predict its magnitude, since $\int_C \mathbf{E} \cdot d\mathbf{s} = 2\pi r E = \mathcal{E}$, which we have already calculated. In this case, the electric field on the circle might look like Fig. 7.18(a) at a particular instant. But if there are other field sources, it could look quite different. If there happened to be a positive and a negative charge located on the axis as shown in Fig. 7.18(b), the electric field in the vicinity of the circle would be the superposition of the electrostatic field of the two charges and the induced electric field.

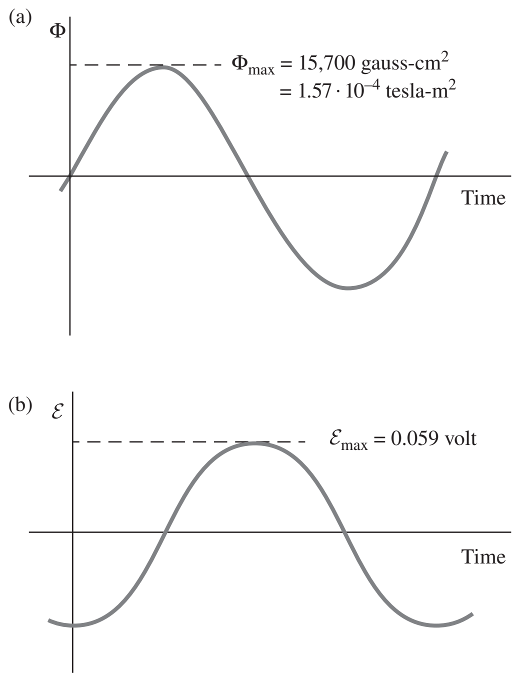


Figure 7.17.
(a) The flux through the circle C. (b) The electromotive force associated with the path C.

A consequence of Faraday's law of induction is that Kirchhoff's loop rule (which states that $\int \mathbf{E} \cdot d\mathbf{s} = 0$ around a closed path) is no longer valid in situations where there is a changing magnetic field. Faraday has taken us beyond the comfortable realm of conservative electric fields. The voltage difference between two points now depends on the path between them. Problem 7.4 provides an instructive example of this fact.

A note on the terminology: the term "*potential* difference" is generally reserved for electrostatic fields, because it is only for such fields that we can uniquely define a potential function ϕ at all points in space, with the property that $\mathbf{E} = -\nabla\phi$. For these fields, the potential difference between points a and b is given by $\phi_b - \phi_a = -\int_a^b \mathbf{E} \cdot d\mathbf{s}$. The term "*voltage* difference" applies to *any* electric field, not necessarily electrostatic, and it is defined similarly as $V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{s}$. If there are changing magnetic fields involved, this line integral will depend on the path between a and b. The voltage difference is what a voltmeter measures, and we can hook up a voltmeter to any type of circuit, of course, no matter what kinds of electric fields it involves. But if there are changing magnetic fields, Problem 7.4 shows that it matters *how* we hook it up. See Romer (1982) for more discussion of this issue.

7.1.3 ■ Motional emf

In the last section, I listed several possible sources of electromotive force, batteries being the most familiar. But I did not mention the commonest one of all: the **generator**. Generators exploit **motional emfs**, which arise when you *move a wire through a magnetic field*. Figure 7.10 suggests a primitive model for a generator. In the shaded region there is a uniform magnetic field **B**, pointing into the page, and the resistor R represents whatever it is (maybe a light bulb or a toaster) we're trying to drive current through. If the entire loop is pulled to the right with speed v, the charges in segment ab experience a magnetic force whose vertical component qvB drives current around the loop, in the clockwise direction. The emf is

$$\mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l} = vBh, \tag{7.11}$$

where h is the width of the loop. (The horizontal segments bc and ad contribute nothing, since the force there is perpendicular to the wire.)

Notice that the integral you perform to calculate $\mathcal{E}$ (Eq. 7.9 or 7.11) is carried out at *one instant of time*—take a "snapshot" of the loop, if you like, and work

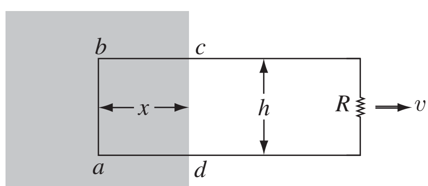


FIGURE 7.10

from that. Thus $d\mathbf{l}$, for the segment ab in Fig. 7.10, points straight up, even though the loop is moving to the right. You can't quarrel with this—it's simply the way emf is *defined*—but it is important to be clear about it.

In particular, although the magnetic force is responsible for establishing the emf, it is *not* doing any work—magnetic forces *never* do work. Who, then, *is* supplying the energy that heats the resistor? *Answer:* The person who's pulling on the loop. With the current flowing, the free charges in segment ab have a vertical velocity (call it $\mathbf{u}$) in addition to the horizontal velocity $\mathbf{v}$ they inherit from the motion of the loop. Accordingly, the magnetic force has a component quB to the left. To counteract this, the person pulling on the wire must exert a force per unit charge

$$f_{\text{pull}} = uB$$

to the *right* (Fig. 7.11). This force is transmitted to the charge by the structure of the wire.

Meanwhile, the particle is actually *moving* in the direction of the resultant velocity $\mathbf{w}$, and the distance it goes is $(h/\cos\theta)$. The work done per unit charge is therefore

$$\int \mathbf{f}_{\text{pull}} \cdot d\mathbf{l} = (uB)\left(\frac{h}{\cos\theta}\right)\sin\theta = vBh = \mathcal{E}$$

($\sin\theta$ coming from the dot product). As it turns out, then, the *work done per unit charge is exactly equal to the emf*, though the integrals are taken along entirely different paths (Fig. 7.12), and completely different forces are involved. To calculate the emf, you integrate around the loop at *one instant*, but to calculate the work done you follow a charge in its journey around the loop; $\mathbf{f}_{\text{pull}}$ contributes nothing to the emf, because it is perpendicular to the wire, whereas $\mathbf{f}_{\text{mag}}$ contributes nothing to work because it is perpendicular to the motion of the charge.[6]

There is a particularly nice way of expressing the emf generated in a moving loop. Let Φ be the flux of $\mathbf{B}$ through the loop:

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}. \tag{7.12}$$

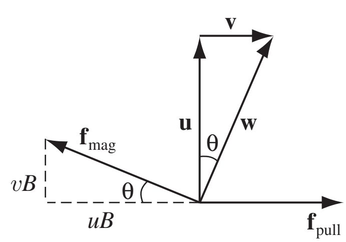
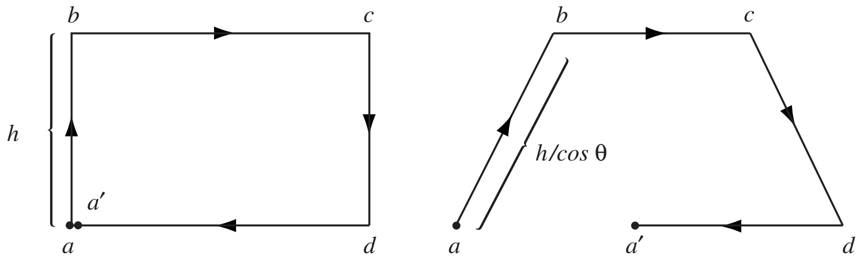


FIGURE 7.11

[6]For further discussion, see E. P. Mosca, *Am. J. Phys.* **42**, 295 (1974).



(a) Integration path for computing $\mathcal{E}$ (follow the wire at one instant of time).

(b) Integration path for calculating work done (follow the charge around the loop).

FIGURE 7.12

For the rectangular loop in Fig. 7.10,

$$\Phi = Bhx.$$

As the loop moves, the flux decreases:

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv.$$

(The minus sign accounts for the fact that dx/dt is negative.) But this is precisely the emf (Eq. 7.11); evidently the emf generated in the loop is minus the rate of change of flux through the loop:

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}.} \tag{7.13}$$

This is the **flux rule** for motional emf.

Apart from its delightful simplicity, the flux rule has the virtue of applying to *non*rectangular loops moving in *arbitrary* directions through *non*uniform magnetic fields; in fact, the loop need not even maintain a fixed shape.

Proof. Figure 7.13 shows a loop of wire at time t, and also a short time dt later. Suppose we compute the flux at time t, using surface $\mathcal{S}$, and the flux at time $t + dt$, using the surface consisting of $\mathcal{S}$ plus the "ribbon" that connects the new position of the loop to the old. The *change* in flux, then, is

$$d\Phi = \Phi(t + dt) - \Phi(t) = \Phi_{\text{ribbon}} = \int_{\text{ribbon}} \mathbf{B} \cdot d\mathbf{a}.$$

Focus your attention on point P: in time dt, it moves to P'. Let $\mathbf{v}$ be the velocity of the *wire*, and $\mathbf{u}$ the velocity of a charge *down* the wire; $\mathbf{w} = \mathbf{v} + \mathbf{u}$ is the resultant

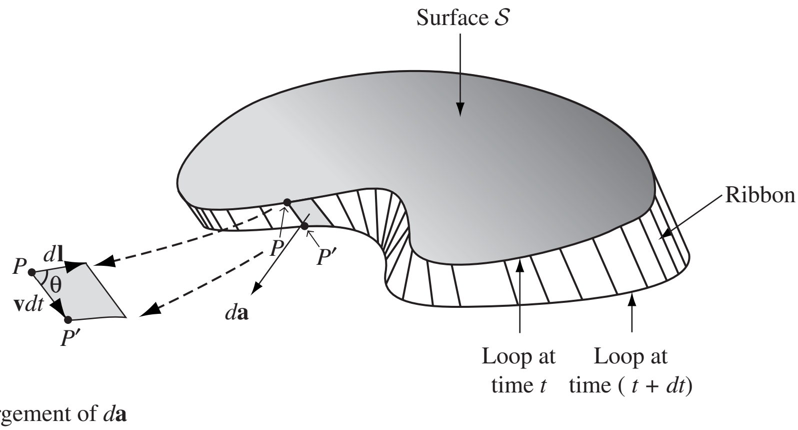


FIGURE 7.13

velocity of a charge at P. The infinitesimal element of area on the ribbon can be written as

$$d\mathbf{a} = (\mathbf{v} \times d\mathbf{l})\, dt$$

(see inset in Fig. 7.13). Therefore

$$\frac{d\Phi}{dt} = \oint \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}).$$

Since $\mathbf{w} = (\mathbf{v} + \mathbf{u})$ and $\mathbf{u}$ is parallel to $d\mathbf{l}$, we can just as well write this as

$$\frac{d\Phi}{dt} = \oint \mathbf{B} \cdot (\mathbf{w} \times d\mathbf{l}).$$

Now, the scalar triple-product can be rewritten:

$$\mathbf{B} \cdot (\mathbf{w} \times d\mathbf{l}) = -(\mathbf{w} \times \mathbf{B}) \cdot d\mathbf{l},$$

so

$$\frac{d\Phi}{dt} = -\oint (\mathbf{w} \times \mathbf{B}) \cdot d\mathbf{l}.$$

But $(\mathbf{w} \times \mathbf{B})$ is the magnetic force per unit charge, $\mathbf{f}_{\text{mag}}$, so

$$\frac{d\Phi}{dt} = -\oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l},$$

and the integral of $\mathbf{f}_{\text{mag}}$ is the emf:

$$\mathcal{E} = -\frac{d\Phi}{dt}. \qquad \square$$

There is a sign ambiguity in the definition of emf (Eq. 7.9): Which *way* around the loop are you supposed to integrate? There is a compensatory ambiguity in the definition of *flux* (Eq. 7.12): Which is the positive direction for $d\mathbf{a}$? In applying

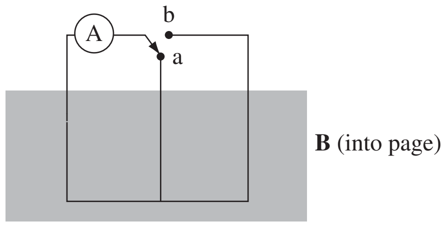


FIGURE 7.14

the flux rule, sign consistency is governed (as always) by your right hand: If your fingers define the positive direction around the loop, then your thumb indicates the direction of $d\mathbf{a}$. Should the emf come out negative, it means the current will flow in the negative direction around the circuit.

The flux rule is a nifty short-cut for calculating motional emfs. It does not contain any new physics—just the Lorentz force law. But it can lead to error or ambiguity if you're not careful. The flux rule assumes you have a single wire loop—it can move, rotate, stretch, or distort (continuously), but beware of switches, sliding contacts, or extended conductors allowing a variety of current paths. A standard "flux rule paradox" involves the circuit in Figure 7.14. When the switch is thrown (from a to b) the flux through the circuit doubles, but there's no motional emf (no conductor moving through a magnetic field), and the ammeter (A) records no current.

Example 7.4. A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform field $\mathbf{B}$, pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk (Fig. 7.15). Find the current in the resistor.

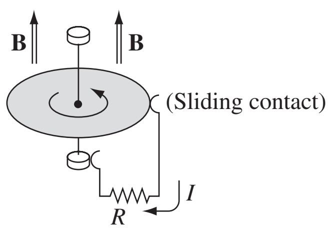


FIGURE 7.15

Solution

The speed of a point on the disk at a distance s from the axis is $v = \omega s$, so the force per unit charge is $\mathbf{f}_{\text{mag}} = \mathbf{v} \times \mathbf{B} = \omega s B \hat{\mathbf{s}}$. The emf is therefore

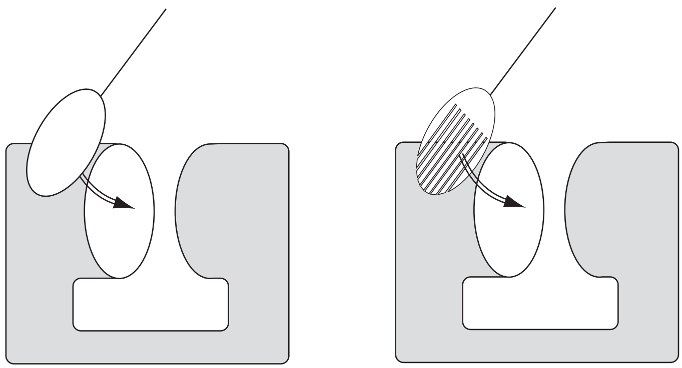
$$\mathcal{E} = \int_0^a f_{\text{mag}}\, ds = \omega B \int_0^a s\, ds = \frac{\omega B a^2}{2},$$

and the current is

$$I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}.$$

Example 7.4 (the **Faraday disk**, or **Faraday dynamo**) involves a motional emf that you can't calculate (at least, not directly) from the flux rule. The flux rule assumes the current flows along a well-defined path, whereas in this example the current spreads out over the whole disk. It's not even clear what the "flux through the circuit" would *mean* in this context.

Even more tricky is the case of **eddy currents**. Take a chunk of aluminum (say), and shake it around in a nonuniform magnetic field. Currents will be generated in the material, and you will feel a kind of "viscous drag"—as though you were pulling the block through molasses (this is the force I called $\mathbf{f}_{\text{pull}}$ in the discussion of motional emf). Eddy currents are notoriously difficult to calculate,[7] but easy and dramatic to demonstrate. You may have witnessed the classic experiment in which an aluminum disk mounted as a pendulum on a horizontal axis swings down and passes between the poles of a magnet (Fig. 7.16a). When it enters the field region it suddenly slows way down. To confirm that eddy currents are responsible, one repeats the demonstration using a disk that has many slots cut in it, to prevent the flow of large-scale currents (Fig. 7.16b). This time the disk swings freely, unimpeded by the field.



(a) (b)

FIGURE 7.16

Problem 7.7 A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart (Fig. 7.17). A resistor R is connected across the rails, and a uniform magnetic field $\mathbf{B}$, pointing into the page, fills the entire region.

[7] See, for example, W. M. Saslow, *Am. J. Phys.*, **60**, 693 (1992).

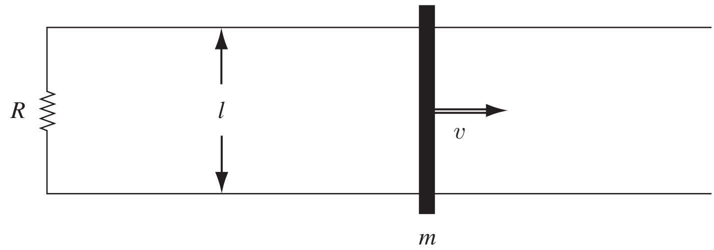


FIGURE 7.17

(a) If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

(b) What is the magnetic force on the bar? In what direction?

(c) If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is its speed at a later time t?

(d) The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0{}^2$. Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0{}^2$.

Problem 7.8 A square loop of wire (side a) lies on a table, a distance s from a very long straight wire, which carries a current I, as shown in Fig. 7.18.

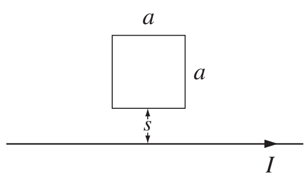


FIGURE 7.18

(a) Find the flux of **B** through the loop.

(b) If someone now pulls the loop directly away from the wire, at speed v, what emf is generated? In what direction (clockwise or counterclockwise) does the current flow?

(c) What if the loop is pulled to the *right* at speed v?

Problem 7.9 An infinite number of different surfaces can be fit to a given boundary line, and yet, in defining the magnetic flux through a loop, $\Phi = \int \mathbf{B} \cdot d\mathbf{a}$, I never specified the particular surface to be used. Justify this apparent oversight.

Problem 7.10 A square loop (side a) is mounted on a vertical shaft and rotated at angular velocity ω (Fig. 7.19). A uniform magnetic field **B** points to the right. Find the $\mathcal{E}(t)$ for this **alternating current** generator.

Problem 7.11 A square loop is cut out of a thick sheet of aluminum. It is then placed so that the top portion is in a uniform magnetic field **B**, and is allowed to fall under gravity (Fig. 7.20). (In the diagram, shading indicates the field region; **B** points into the page.) If the magnetic field is 1 T (a pretty standard laboratory field), find the terminal velocity of the loop (in m/s). Find the velocity of the loop as a function of time. How long does it take (in seconds) to reach, say, 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit? [*Note:* The dimensions of the loop cancel out; determine the actual *numbers*, in the units indicated.]

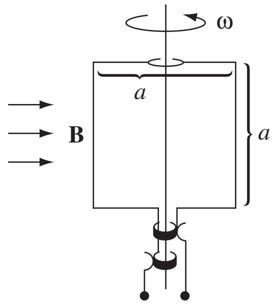


FIGURE 7.19

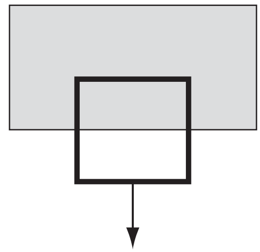


FIGURE 7.20

7.2 ■ ELECTROMAGNETIC INDUCTION

7.2.1 ■ Faraday's Law

In 1831 Michael Faraday reported on a series of experiments, including three that (with some violence to history) can be characterized as follows:

Experiment 1. He pulled a loop of wire to the right through a magnetic field (Fig. 7.21a). A current flowed in the loop.

Experiment 2. He moved the *magnet* to the *left*, holding the loop still (Fig. 7.21b). Again, a current flowed in the loop.

Experiment 3. With both the loop and the magnet at rest (Fig. 7.21c), he changed the *strength* of the field (he used an electromagnet, and varied the current in the coil). Once again, current flowed in the loop.

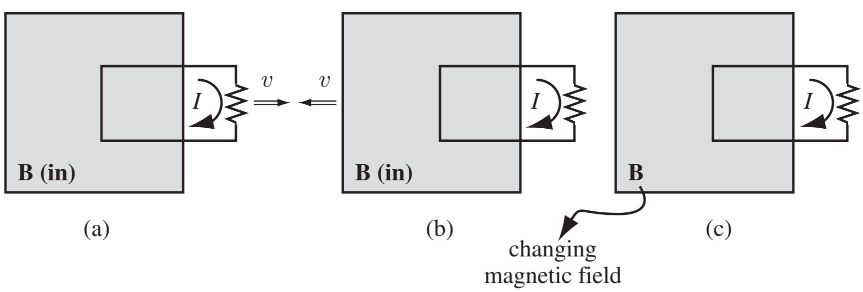


FIGURE 7.21

The first experiment, of course, is a straightforward case of motional emf; according to the flux rule:

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

I don't think it will surprise you to learn that exactly the same emf arises in Experiment 2—all that really matters is the *relative* motion of the magnet and the loop. Indeed, in the light of special relativity it *has* to be so. But Faraday knew nothing of relativity, and in classical electrodynamics this simple reciprocity is a remarkable coincidence. For if the *loop* moves, it's a *magnetic* force that sets up the emf, but if the loop is *stationary*, the force *cannot* be magnetic—stationary charges experience no magnetic forces. In that case, what *is* responsible? What sort of field exerts a force on charges at rest? Well, *electric* fields do, of course, but in this case there doesn't seem to be any electric field in sight.

Faraday had an ingenious inspiration:

A changing magnetic field induces an electric field.

It is this induced[8] electric field that accounts for the emf in Experiment 2.[9] Indeed, if (as Faraday found empirically) the emf is again equal to the rate of change of the flux,

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}, \tag{7.14}$$

then **E** is related to the change in **B** by the equation

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}. \tag{7.15}$$

This is **Faraday's law**, in integral form. We can convert it to differential form by applying Stokes' theorem:

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.} \tag{7.16}$$

[8]"Induce" is a subtle and slippery verb. It carries a faint odor of *causation* ("*pro*duce" would make this explicit) without quite committing itself. There is a sterile ongoing debate in the literature as to whether a changing magnetic field should be regarded as an independent "source" of electric fields (along with electric charge)—after all, the magnetic field *itself* is due to electric currents. It's like asking whether the postman is the "source" of my mail. Well, sure—he delivered it to my door. On the other hand, Grandma wrote the letter. Ultimately, ρ and **J** are the sources of *all* electromagnetic fields, and a changing magnetic field merely delivers electromagnetic news from currents elsewhere. But it is often convenient to think of a changing magnetic field "producing" an electric field, and it won't hurt you as long as you understand that this is the condensed version of a more complicated story. For a nice discussion, see S. E. Hill, *Phys. Teach*. **48**, 410 (2010).

[9]You might argue that the magnetic field in Experiment 2 is not really *changing*—just *moving*. What I mean is that if you sit at a *fixed location*, the field you experience changes as the magnet passes by.

Note that Faraday's law reduces to the old rule $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ (or, in differential form, $\nabla \times \mathbf{E} = \mathbf{0}$) in the static case (constant $\mathbf{B}$) as, of course, it should.

In Experiment 3, the magnetic field changes for entirely different reasons, but according to Faraday's law an electric field will again be induced, giving rise to an emf $-d\Phi/dt$. Indeed, one can subsume all three cases (and for that matter any combination of them) into a kind of **universal flux rule**:

> **Whenever (and for whatever reason) the magnetic flux through a loop changes, an emf**
>
> $$\mathcal{E} = -\frac{d\Phi}{dt} \tag{7.17}$$
>
> **will appear in the loop**.

Many people call *this* "Faraday's law." Maybe I'm overly fastidious, but I find this confusing. There are really *two* totally different mechanisms underlying Eq. 7.17, and to identify them both as "Faraday's law" is a little like saying that because identical twins look alike we ought to call them by the same name. In Faraday's first experiment it's the Lorentz force law at work; the emf is *magnetic*. But in the other two it's an *electric* field (induced by the changing magnetic field) that does the job. Viewed in this light, it is quite astonishing that all three processes yield the same formula for the emf. In fact, it was precisely this "coincidence" that led Einstein to the special theory of relativity—he sought a deeper understanding of what is, in classical electrodynamics, a peculiar accident. But that's a story for Chapter 12. In the meantime, I shall reserve the term "Faraday's law" for electric fields induced by changing magnetic fields, and I do *not* regard Experiment 1 as an instance of Faraday's law.

Example 7.5. A long cylindrical magnet of length L and radius a carries a uniform magnetization $\mathbf{M}$ parallel to its axis. It passes at constant velocity v through a circular wire ring of slightly larger diameter (Fig. 7.22). Graph the emf induced in the ring, as a function of time.

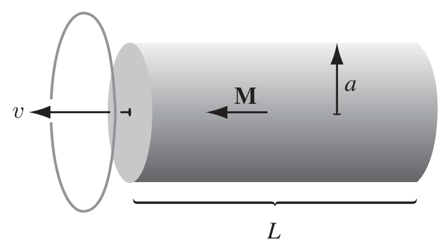


FIGURE 7.22

Solution

The magnetic field is the same as that of a long solenoid with surface current $\mathbf{K}_b = M\,\hat{\boldsymbol{\phi}}$. So the field inside is $\mathbf{B} = \mu_0\mathbf{M}$, except near the ends, where it starts to spread out. The flux through the ring is zero when the magnet is far away; it builds up to a maximum of $\mu_0 M\pi a^2$ as the leading end passes through; and it drops back to zero as the trailing end emerges (Fig. 7.23a). The emf is (minus) the derivative of Φ with respect to time, so it consists of two spikes, as shown in Fig. 7.23b.

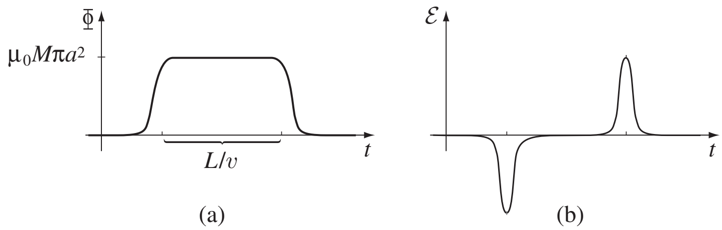


FIGURE 7.23

Keeping track of the *signs* in Faraday's law can be a real headache. For instance, in Ex. 7.5 we would like to know which *way* around the ring the induced current flows. In principle, the right-hand rule does the job (we called Φ positive to the left, in Fig. 7.22, so the positive direction for current in the ring is counterclockwise, as viewed from the left; since the first spike in Fig. 7.23b is *negative*, the first current pulse flows *clockwise*, and the second counterclockwise). But there's a handy rule, called **Lenz's law**, whose sole purpose is to help you get the directions right:[10]

Nature abhors a change in flux.

The induced current will flow in such a direction that the flux *it* produces tends to cancel the change. (As the front end of the magnet in Ex. 7.5 enters the ring, the flux increases, so the current in the ring must generate a field to the *right*—it therefore flows *clockwise*.) Notice that it is the *change* in flux, not the flux itself, that nature abhors (when the tail end of the magnet exits the ring, the flux *drops*, so the induced current flows *counterclockwise*, in an effort to restore it). Faraday induction is a kind of "inertial" phenomenon: A conducting loop "likes" to maintain a constant flux through it; if you try to *change* the flux, the loop responds by sending a current around in such a direction as to frustrate your efforts. (It doesn't *succeed* completely; the flux produced by the induced current is typically only a tiny fraction of the original. All Lenz's law tells you is the *direction* of the flow.)

[10] Lenz's law applies to *motional* emfs, too, but for them it is usually easier to get the direction of the current from the Lorentz force law.

Example 7.6. The "jumping ring" demonstration. If you wind a solenoidal coil around an iron core (the iron is there to beef up the magnetic field), place a metal ring on top, and plug it in, the ring will jump several feet in the air (Fig. 7.24). Why?

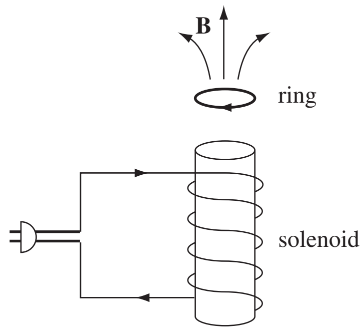


FIGURE 7.24

Solution

Before you turned on the current, the flux through the ring was *zero*. *Afterward* a flux appeared (upward, in the diagram), and the emf generated in the ring led to a current (in the ring) which, according to Lenz's law, was in such a direction that *its* field tended to cancel this new flux. This means that the current in the loop is *opposite* to the current in the solenoid. And opposite currents repel, so the ring flies off.[11]

Problem 7.12 A long solenoid, of radius a, is driven by an alternating current, so that the field inside is sinusoidal: $\mathbf{B}(t) = B_0 \cos(\omega t)\,\hat{\mathbf{z}}$. A circular loop of wire, of radius $a/2$ and resistance R, is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time.

Problem 7.13 A square loop of wire, with sides of length a, lies in the first quadrant of the xy plane, with one corner at the origin. In this region, there is a nonuniform time-dependent magnetic field $\mathbf{B}(y, t) = ky^3t^2\,\hat{\mathbf{z}}$ (where k is a constant). Find the emf induced in the loop.

Problem 7.14 As a lecture demonstration a short cylindrical bar magnet is dropped down a vertical aluminum pipe of slightly larger diameter, about 2 meters long. It takes several seconds to emerge at the bottom, whereas an otherwise identical piece of *unmagnetized* iron makes the trip in a fraction of a second. Explain why the magnet falls more slowly.[12]

[11]For further discussion of the jumping ring (and the related "floating ring"), see C. S. Schneider and J. P. Ertel, *Am. J. Phys.* **66**, 686 (1998); P. J. H. Tjossem and E. C. Brost, *Am. J. Phys.* **79**, 353 (2011).
[12]For a discussion of this amazing demonstration see K. D. Hahn et al., *Am. J. Phys.* **66**, 1066 (1998) and G. Donoso, C. L. Ladera, and P. Martin, *Am. J. Phys.* **79**, 193 (2011).

7.2.2 ■ The Induced Electric Field

Faraday's law generalizes the electrostatic rule $\nabla \times \mathbf{E} = \mathbf{0}$ to the time-dependent régime. The *divergence* of $\mathbf{E}$ is still given by Gauss's law ($\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$). If $\mathbf{E}$ is a *pure* Faraday field (due exclusively to a changing $\mathbf{B}$, with $\rho = 0$), then

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

This is mathematically identical to magnetostatics,

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Conclusion: Faraday-induced electric fields are determined by $-(\partial \mathbf{B}/\partial t)$ in exactly the same way as magnetostatic fields are determined by $\mu_0 \mathbf{J}$. The analog to Biot-Savart is[13] is

$$\mathbf{E} = -\frac{1}{4\pi}\int \frac{(\partial \mathbf{B}/\partial t) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{\mathbf{B} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau, \tag{7.18}$$

and if symmetry permits, we can use all the tricks associated with Ampère's law in integral form ($\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$), only now it's *Faraday's* law in integral form:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}. \tag{7.19}$$

The rate of change of (magnetic) flux through the Amperian loop plays the role formerly assigned to $\mu_0 I_{\text{enc}}$.

Example 7.7. A uniform magnetic field $\mathbf{B}(t)$, pointing straight up, fills the shaded circular region of Fig. 7.25. If $\mathbf{B}$ is changing with time, what is the induced electric field?

Solution

$\mathbf{E}$ points in the circumferential direction, just like the *magnetic* field inside a long straight wire carrying a uniform *current* density. Draw an Amperian loop of radius s, and apply Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = -\pi s^2 \frac{dB}{dt}.$$

Therefore

$$\mathbf{E} = -\frac{s}{2}\frac{dB}{dt}\,\hat{\boldsymbol{\phi}}.$$

If $\mathbf{B}$ is *increasing*, $\mathbf{E}$ runs *clockwise*, as viewed from above.

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[13]Magnetostatics holds only for time-independent currents, but there is no such restriction on $\partial \mathbf{B}/\partial t$.

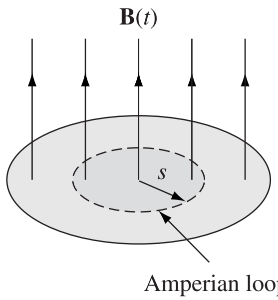

FIGURE 7.25

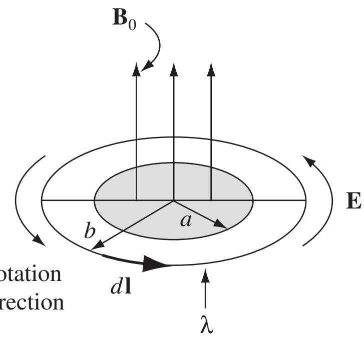

FIGURE 7.26

Example 7.8. A line charge λ is glued onto the rim of a wheel of radius b, which is then suspended horizontally, as shown in Fig. 7.26, so that it is free to rotate (the spokes are made of some nonconducting material—wood, maybe). In the central region, out to radius a, there is a uniform magnetic field $\mathbf{B}_0$, pointing up. Now someone turns the field off. What happens?

Solution

The changing magnetic field will induce an electric field, curling around the axis of the wheel. This electric field exerts a force on the charges at the rim, and the wheel starts to turn. According to Lenz's law, it will rotate in such a direction that *its* field tends to restore the upward flux. The motion, then, is counterclockwise, as viewed from above.

Faraday's law, applied to the loop at radius b, says

$$\oint \mathbf{E}\cdot d\mathbf{l} = E(2\pi b) = -\frac{d\Phi}{dt} = -\pi a^2\frac{dB}{dt}, \quad \text{or} \quad \mathbf{E} = -\frac{a^2}{2b}\frac{dB}{dt}\,\hat{\boldsymbol{\phi}}.$$

The torque on a segment of length $d\mathbf{l}$ is $(\mathbf{r}\times\mathbf{F})$, or $b\lambda E\,dl$. The total torque on the wheel is therefore

$$N = b\lambda\left(-\frac{a^2}{2b}\frac{dB}{dt}\right)\oint dl = -b\lambda\pi a^2\frac{dB}{dt},$$

and the angular momentum imparted to the wheel is

$$\int N\,dt = -\lambda\pi a^2 b\int_{B_0}^{0} dB = \lambda\pi a^2 b B_0.$$

It doesn't matter how quickly or slowly you turn off the field; the resulting angular velocity of the wheel is the same regardless. (If you find yourself wondering where the angular momentum *came* from, you're getting ahead of the story! Wait for the next chapter.)

Note that it's the *electric* field that did the rotating. To convince you of this, I deliberately set things up so that the *magnetic* field is *zero* at the location of the charge. The experimenter may tell you she never put in any electric field—all she did was switch off the magnetic field. But when she did that, an electric field automatically appeared, and it's this electric field that turned the wheel.

I must warn you, now, of a small fraud that tarnishes many applications of Faraday's law: Electromagnetic induction, of course, occurs only when the magnetic fields are *changing*, and yet we would like to use the apparatus of magneto*statics* (Ampère's law, the Biot-Savart law, and the rest) to *calculate* those magnetic fields. Technically, any result derived in this way is only approximately correct. But in practice the error is usually negligible, unless the field fluctuates extremely rapidly, or you are interested in points very far from the source. Even the case of a wire snipped by a pair of scissors (Prob. 7.18) is *static enough* for Ampère's law to apply. This régime, in which magnetostatic rules can be used to calculate the magnetic field on the right hand side of Faraday's law, is called **quasistatic**. Generally speaking, it is only when we come to electromagnetic waves and radiation that we must worry seriously about the breakdown of magnetostatics itself.

Example 7.9. An infinitely long straight wire carries a slowly varying current $I(t)$. Determine the induced electric field, as a function of the distance s from the wire.[14]

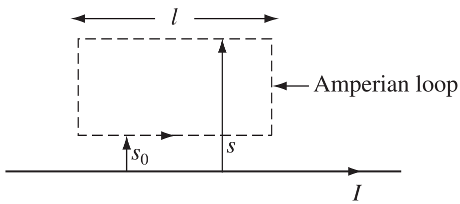

FIGURE 7.27

Solution

In the quasistatic approximation, the magnetic field is $(\mu_0 I/2\pi s)$, and it circles around the wire. Like the **B**-field of a solenoid, **E** here runs parallel to the axis. For the rectangular "Amperian loop" in Fig. 7.27, Faraday's law gives:

$$\begin{aligned} \oint \mathbf{E}\cdot d\mathbf{l} &= E(s_0)l - E(s)l = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} \\ &= -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}\int_{s_0}^{s}\frac{1}{s'}\,ds' = -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}(\ln s - \ln s_0). \end{aligned}$$

[14]This example is artificial, and not just in the obvious sense of involving infinite wires, but in a more subtle respect. It assumes that the current is the same (at any given instant) all the way down the line. This is a safe assumption for the *short* wires in typical electric circuits, but not for *long* wires (**transmission lines**), unless you supply a distributed and synchronized driving mechanism. But never mind—the problem doesn't inquire how you would *produce* such a current; it only asks what *fields* would result if you *did*. Variations on this problem are discussed by M. A. Heald, *Am. J. Phys.* **54**, 1142 (1986).

Thus

$$\mathbf{E}(s) = \left[\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln s + K \right] \hat{\mathbf{z}}, \tag{7.20}$$

where K is a constant (that is to say, it is independent of s—it might still be a function of t). The actual *value* of K depends on the whole history of the function $I(t)$—we'll see some examples in Chapter 10.

Equation 7.20 has the peculiar implication that E blows up as s goes to infinity. *That* can't be true ... What's gone wrong? *Answer:* We have overstepped the limits of the quasistatic approximation. As we shall see in Chapter 9, electromagnetic "news" travels at the speed of light, and at large distances **B** depends not on the current *now*, but on the current *as it was* at some earlier time (indeed, a whole *range* of earlier times, since different points on the wire are different distances away). If τ is the time it takes I to change substantially, then the quasistatic approximation should hold only for

$$s \ll c\tau, \tag{7.21}$$

and hence Eq. 7.20 simply does not apply, at extremely large s.

Problem 7.15 A long solenoid with radius a and n turns per unit length carries a time-dependent current $I(t)$ in the $\hat{\boldsymbol{\phi}}$ direction. Find the electric field (magnitude and direction) at a distance s from the axis (both inside and outside the solenoid), in the quasistatic approximation.

Problem 7.16 An alternating current $I = I_0 \cos(\omega t)$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a.

(a) In what *direction* does the induced electric field point (radial, circumferential, or longitudinal)?

(b) Assuming that the field goes to zero as $s \to \infty$, find $\mathbf{E}(s, t)$.[15]

Problem 7.17 A long solenoid of radius a, carrying n turns per unit length, is looped by a wire with resistance R, as shown in Fig. 7.28.

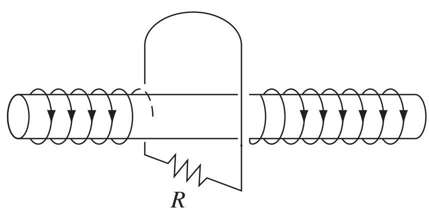

FIGURE 7.28

[15]This is not at all the way electric fields *actually* behave in coaxial cables, for reasons suggested in the previous footnote. See Sect. 9.5.3, or J. G. Cherveniak, *Am. J. Phys.*, **54**, 946 (1986), for a more realistic treatment.

(a) If the current in the solenoid is increasing at a constant rate ($dI/dt = k$), what current flows in the loop, and which way (left or right) does it pass through the resistor?

(b) If the current I in the solenoid is constant but the solenoid is pulled out of the loop (toward the left, to a place far from the loop), what total charge passes through the resistor?

Problem 7.18 A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I (Fig. 7.29). Now someone cuts the wire, so I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows? If you don't like the scissors model, turn the current down *gradually*:

$$I(t) = \begin{cases} (1-\alpha t)I, & \text{for } 0 \le t \le 1/\alpha, \\ 0, & \text{for } t > 1/\alpha. \end{cases}$$

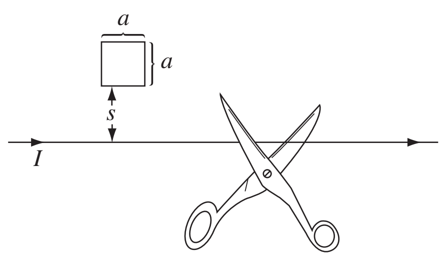

FIGURE 7.29

Problem 7.19 A toroidal coil has a rectangular cross section, with inner radius a, outer radius $a + w$, and height h. It carries a total of N tightly wound turns, and the current is increasing at a constant rate ($dI/dt = k$). If w and h are both much less than a, find the electric field at a point z above the center of the toroid. [*Hint:* Exploit the analogy between Faraday fields and magnetostatic fields, and refer to Ex. 5.6.]

Problem 7.20 Where is $\partial \mathbf{B}/\partial t$ nonzero, in Figure 7.21(b)? Exploit the analogy between Faraday's law and Ampère's law to sketch (qualitatively) the electric field.

Problem 7.21 Imagine a uniform magnetic field, pointing in the z direction and filling all space ($\mathbf{B} = B_0\,\hat{\mathbf{z}}$). A positive charge is at rest, at the origin. Now somebody turns off the magnetic field, thereby inducing an electric field. In what direction does the charge move?[16]